

Dynamics of threshold network on non-trivial distribution degree

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Abstract. The dynamics of a threshold network (TN) with thermal noise on scale-free, random-graph, and small-world topologies are considered herein. The present analytical study clarifies that there is no phase transition independent of network structure if temperature $T = 0$, threshold $h = 0$ and the probability distribution degree $P(k)$ satisfies $P(0) = D = 0$. The emergence of phase transition involving three parameters, T, h and D is also investigated. We find that a TN with moderate thermal noise extends the regime of ordered dynamics, compared to a TN in the $T = 0$ regime or a Random Boolean Network (RBN). A TN can be continuously reduced to an expression of RBN in the infinite T limit.

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1 Introduction

Recently, considerable interest has been expressed in the formation of complex networks, in their connectivity and particularly in the dynamics of these networks [1–3]. The term ‘connectivity’ is defined as a number of links which are connected to a given node. Many real networks have a local structure that differ from that of random networks having finite connectivity. Examples of such networks include the WWW [4], Internet [5], the gene regulatory network [6], and protein networks [7,8]. These networks exhibit a power-law distribution degree, corresponding to the distribution of connectivities, which leads to the construction of scale-free models [1]. A scale-free model has a property whereby a small fraction of the nodes, often called a hub, is highly connected, whereas the majority of the nodes have low connectivity. Although the topological properties of these models have been studied in detail, the relationship between dynamics of cooperative processes with some interaction rules and topological structure has been studied to a lesser extent.

In the present paper, we explore the dynamics of a threshold network (TN), which is first investigated as diluted, asymmetric spin-glass models [12], and asymmetric neural networks [13,14]. A network of such interaction rules with non-trivial random topology is of interest as a theoretical framework for investigating the basic properties of natural dynamical networks. Moreover, applications such as improved information processing in neural

networks by controlling the noise of interaction rules and network topology or capturing the essentials of gene regulatory networks in a biological system with a given hierarchical topology may be derived from such a theoretical ‘toy model’. The TN is closely related to the Random Boolean Network (RBN) first introduced by Kauffman [9,10] to model rule-based gene regulatory networks. We herein introduce a thermal noise parameter T , by which a TN with infinite T is mapped to an RBN in the mean field theory. Therefore, the behavior of the TN in the high T regime (i.e. $\beta = T^{-1} \approx 0$) can be interpreted as a perturbation of the RBN. For the RBN model, the state $\sigma_i \in \{0, 1\}$ of a network node i is a logical function of the states of k_i other nodes chosen at random. The logical functions are chosen at random, with a suitably biased probability. In mean field theory, the RBN model shows a phase transition with respect to the number of inputs per node K at a critical average connectivity $\langle k_c \rangle = [2p(1-p)]^{-1}$ from an ordered to a chaotic state, where p is the probability that the randomly chosen output of node i is unity. In the chaotic state regime ($K > \langle k_c \rangle$), a small perturbation in the initial state propagates across the entire system, whereas all perturbation in the initial state dies out in the ordered state regime ($K < \langle k_c \rangle$). The dynamics of the RBN on a scale-free model has recently been investigated [11]. However, there seems to be no non-trivial difference between the dynamics of the scale-free model and that of the traditional RBN because the output state of the nodes in the RBN is independent of the distribution degree, whereas in the TN the output state of the

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nodes depends on the number of inputs. Other studies have shown that the critical average connectivity in a random threshold network (in which linked nodes are randomly chosen) is below $\langle k_c \rangle = 2$, which is in contrast to the RBN, in which the critical average connectivity is always $\langle k_c \rangle \geq 2$ [15]. In the following, we measure the propagation of a small perturbation in the initial state on the scale-free, random-graph (Poisson distribution) and small-world models, which allows us to evaluate the effect of thermal noise and topological structure on the dynamics of the TN.

2 Threshold network

We consider a network of N randomly interconnected binary nodes of state $\sigma_i = \pm 1$. In this model, the probability distribution of outputs depends strongly on its degree parameters. At time t , the fields $f_i(t)$ of node i are computed:

$$f_i(t) = \sum_{j=1}^N c_{ij} \sigma_j(t) + h, \quad (1)$$

where the threshold parameter h is a natural number or zero. The threshold characteristics often provide an intrinsic feature of the network, i.e. the threshold for neuron firing in the neural network model. The interaction weight c_{ij} takes a discrete value $c_{ij} = \pm 1$, with equal possibility. In order to avoid *self connection*, we assume $c_{ii} = 0$. If node i does not receive a signal from j , then node j also has $c_{ij} = 0$. The degree k_i of node i is defined as $k_i = \sum_j^N |c_{ij}|$. The interaction weight c_{ij} is chosen from all nodes but follows the probability distribution degree $P(k)$. For each node i , the state of the mode at time $t + 1$ is a function of the inputs it receives from other nodes at time t

$$\sigma_i(t+1) = \begin{cases} 1, & \text{with probability } G(f_i(t)), \\ -1, & \text{with probability } G(-f_i(t)), \end{cases} \quad (2)$$

where $G(x) = [1 + \exp(-2x/T)]^{-1}$. We now introduce a parameter T as thermal noise (temperature) of the system to control the dependence of node states on their fields. The N network nodes are idealized by synchronized updating. Generalizing the so-called annealing approximation method introduced by Derrida and Pomeau [16], we approach the system analytically in the limit of infinite N . The normalized overlap function x obey the equation

$$x(t+1) = F(x(t)), \quad (3)$$

where the mapping function $F(x)$ is given by

$$F(x) \equiv 1 - \sum_{k=1}^{\tilde{K}} p_s(k, h, T) P(k) (1 - x^k), \quad (4)$$

with cutoff \tilde{K} . The normalized overlap function $x(t) = 1 - d(\boldsymbol{\sigma}(t), \boldsymbol{\sigma}'(t))/N$ is obtained by subtracting the normalized Hamming distance from unity, where the Hamming distance $d(\boldsymbol{\sigma}(t), \boldsymbol{\sigma}'(t))$ denotes the overlap between

two time independent distinctive configurations. In the $N \rightarrow \infty$ limit, the function x represents the probability for two arbitrary configurations to be equal. The possibility p_s is the output state reversal obtained by changing the state of a single input j . Using combinatorial methods [15], the stochastic distribution of p_s is derived as follows:

$$p_s(k, h, T) = \frac{1}{k \cdot 2^k} \sum_{i=0}^k \binom{k}{i} \times \left(G(k+h-2i)(k-(k-i)G(k+h-2-2i)) - iG(k+h+2-2i) \right) + (1 - G(k+h-2i)) \times \left((k-i)G(k+h-2-2i) + iG(k+h+2-2i) \right). \quad (5)$$

The stationary value of the overlap function x^* is obtained by means of a fixed point equation $x = F(x)$. The existence of a phase transition depends uniquely on the nature of the fixed point $x = 1$. If the point is attractive, two initial configurations differing by an infinitesimal fraction of nodes will become almost identical, whereas if the point is repulsive, the configurations will produce diverging trajectories. A possible critical point is then determined by the following equation:

$$\left. \frac{dF}{dx} \right|_{x=1} = 1. \quad (6)$$

Let us recall briefly the sufficient conditions for an attractive fixed point x^* . Here, equation (4) must satisfy (a) $dF/dx \geq 0$ for all $0 \leq x \leq 1$, (b) $F(0) > 0$ and (c) $dF/dx|_{x=1} > 1$. For $T = 0$, p_s is given by,

$$p_s(k, h) = \begin{cases} \binom{k}{(k+h)/2} / 2^k, & \text{for } k+h \text{ is even,} \\ p_s(k-1, h), & \text{for } k+h \text{ is odd.} \end{cases} \quad (7)$$

Moreover, for $h = 0$, (a), (b) are satisfied, and (c) is also satisfied for any $P(k)$ if $P(0) = 0$. The proof of (c) is straightforward, and is given by showing that $kp_s(k) - 1 \geq 0$ for $\forall k \geq 1$, using asymptotically $p_s(k) \sim k^{-1/2}$ for large k . The fixed point $x^* = 1$ is always repulsive, and another fixed point $x^* < 1$ is stable. Thus, independent of $P(k)$ for $k \geq 1$, there is no phase transition and a small perturbation/damage could propagate over the entire system. In the following, we discuss the effect of the distribution degree on the network dynamics.

3 Scale-free model

We first introduce the scale-free distribution degree given by,

$$P(k) = \begin{cases} D, & k = 0, \\ (1-D)\eta(\gamma, \tilde{K})^{-1} k^{-\gamma}, & 1 \leq k \leq \tilde{K}, \end{cases} \quad (8)$$

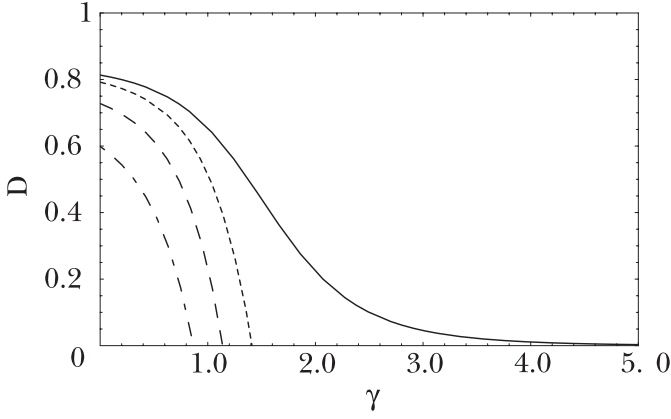


Fig. 1. Curve of critical value D_c as a function of γ having cutoff $\tilde{K} = 100$ and thresholds $h = 0$ (solid line), $h = 3$ (dotted line), $h = 6$ (dashed line), and $h = 9$ (dot-dashed line).

where $\eta(\gamma, \tilde{K}) = \sum_{k=1}^{\tilde{K}} k^{-\gamma}$, $0 \leq D \leq 1$ and γ is usually referred to as a scale-free exponent. Substituting equation (8) into equation (6) with $T = h = 0$, we obtain the critical condition for D

$$D_c = \frac{A(\gamma) - 1}{A(\gamma)}, \quad A(\gamma) = \eta(\gamma, \tilde{K})^{-1} \sum_{k=1}^{\tilde{K}} p_s(k, 0) k^{1-\gamma}. \quad (9)$$

Since $1 - p_s(k, 0) \geq 0$ for $\forall k \geq 1$, the critical average connectivity $\langle k_c \rangle = \eta(\gamma - 1, \tilde{K}) / (A(\gamma) * \eta(\gamma, \tilde{K}))$ has a minimum value in the limit of infinite γ , i.e. $\langle k_c \rangle = 1$ is realized when $D_c = 0$ (see Fig. 1). In Figure 2, we plot the stationary overlap function x^* for various threshold parameters provided that T and D are set to zero. The overlap function for $h = 0$ is shown to asymptotically approach $x^* \rightarrow 1$ with increasing γ , i.e. there is no phase transition until average the connectivity reaches $\langle k \rangle = 1$. This result is different from the well known chaotic to ordered phase transition observed in RBN [9]. For the finite threshold h , we observe that the equilibrium overlap x^* increases as h increases, which suggests that the threshold eliminates the memory of the initial condition and the two different initial conditions become identical. For comparison, the curve of x^* for the RBN is also shown. For the low- γ regime, a perturbation is suppressed to a greater degree in the TN with zero-threshold than in the RBN. The opposite is observed for high- γ regime. The critical point at which the strength of the perturbation reverse is given by $\gamma_s \approx 1.754$ in this model. This result supports the findings of the numerical simulation shown in Figure 8 of [15].

The inset of Figure 3a shows the curve of the critical scale-free exponent γ_c as a function of T for $h = D = 0$, which distinguishes the ordered state from the chaotic state. For finite T , $dF/dx|_{x=1}$ in the large γ limit is

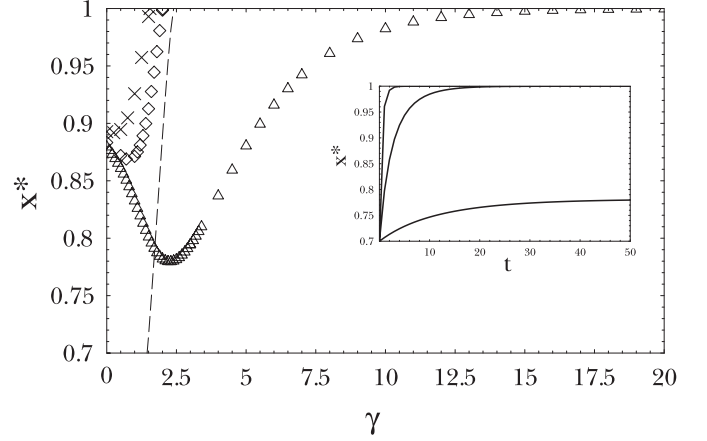


Fig. 2. Normalized overlap function x in equilibrium as a function of γ with $h = 0$ (Δ), $h = 1$ (\diamond), and $h = 2$ (\times). The dashed line denotes RBN with $p = 1/2$. The cutoff $\tilde{K} = 100$, and the temperature $T = 0$. The critical value γ_c does not exist (infinity) for $h = 0$, but is 2.007 for $h = 1$ and 1.551 for $h = 2$; i.e., $\langle k_c \rangle = 1$ ($h = 0$), $\langle k_c \rangle = 3.140$ ($h = 1$), and $\langle k_c \rangle = 6.800$ ($h = 2$). The inset shows the time evolution of the overlap function $x(t)$ with $x(0) = 0.7$ and $\gamma = 2.5$ for each threshold.

given by,

$$\frac{dF}{dx}|_{x=1-} = \frac{1}{(1 + \exp(-2/T))^2} + \frac{1}{(1 + \exp(2/T))^2} + O(1/N). \quad (10)$$

The existence of infinitesimal T which satisfies $dF/dx|_{x=1} < 1$ results in a critical temperature $T_c = 0$. The value of γ_c diverges at zero temperature, as shown in the inset of Figure 3a. An interesting feature of the $\langle k_c \rangle$ curve is that it has a maximum $\langle k_c \rangle$ value at moderate T (Fig. 3a), which indicates that the thermal noise can control the chaotic state and bring it into an ordered state. In the infinite T limit, we obtain $p_s = 1/2$ which is identical to that of the RBN with $p = 1/2$. By substituting $p_s = 1/2$ into equation (6), the critical condition of the RBN is obtained.

$$\frac{\eta(-1 + \gamma, \tilde{K})}{2\eta(\gamma, \tilde{K})} = 1. \quad (11)$$

Since the average connectivity is given by $\langle k \rangle = \eta(-1 + \gamma, \tilde{K}) / \eta(\gamma, \tilde{K})$, we obtain $\langle k_c \rangle = 2$ from equation (11). In the $\tilde{K} \rightarrow \infty$ limit, equation (11) is represented by the Riemann ζ function, which gives $\gamma_c \approx 2.4788$. Note that $\langle k_c \rangle$ is independent of \tilde{K} and γ , as well as all topological structures, in the infinite T limit.

4 Random-graph model

In a random-graph model, the distribution degree follows $P(k) = e^{-K_p} K_p^k / k!$, where the parameter K_p is the average connectivity of the network. By solving equation (6),

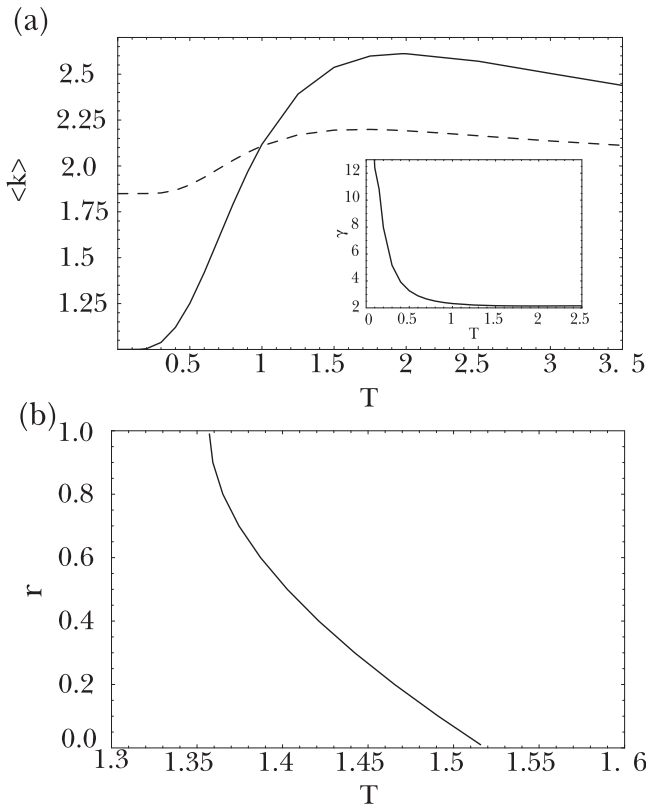


Fig. 3. (a) Critical average connectivity $\langle k_c \rangle$ curve of the scale-free model (solid line) and random-graph model (dotted line) having $h = 0$ and $\tilde{K} = 100$. The scale-free model has a maximum $\langle k_c \rangle = 2.611$ at $T = 1.970$, and the random-graph model has a maximum $\langle k_c \rangle = 2.199$ at $T = 1.683$. Both models subsequently converge to $\langle k_c \rangle = 2$ in the infinite T limit. The inset shows the $\gamma - T$ diagram of the scale-free model. In the chaotic state regime ($\gamma < \gamma_c$), the overlap function in the stationary state x^* satisfies $x^* < 1$. In the ordered state regime ($\gamma > \gamma_c$), $x^* = 1$. (b) Critical rewiring probability r_c versus T for the small-world model with average connectivity $2K_w = 2$. This r_c curve also divides the system into two regimes, the chaotic state ($r < r_c$) and the ordered state ($r > r_c$).

the critical average connectivity $\langle k_c \rangle = K_p = 1.849$ at $T = 0$ is obtained as being identical to that reported in [15]. The distribution degree with zero connectivity is $P(0) = 0.157$. Comparing the T dependence on $\langle k_c \rangle$ with that of the scale-free model, the behavior of the random-graph model is found to be less sensitive to T than that of scale-free (Fig. 3a). However, both results have a common feature in that moderate thermal noise can extend the regime of ordered dynamics in TN to higher average network connectivity, compared to the zero-temperature and high-temperature regimes. In both models, the critical average connectivity goes to $\langle k_c \rangle = 2$ in the infinite T limit.

5 Small-world model

In a small-world model [2], we start with an ordered state, in which each node has the same connectivity $2K_w$.

By randomly rewiring each link between nearest-neighbor nodes with probability r , we introduce non-uniform connectivity into the network, while maintaining a fixed integer average connectivity $\langle k \rangle = 2K_w$. The probability distribution degree is given by [17]

$$P(k) = \sum_{n=0}^{f(k, K_w)} \binom{K_w}{n} (1-r)^n r^{K_w-n} \frac{(rK_w)^{k-K_w-n}}{(k-K_w-n)!} e^{-rK_w}, \quad (12)$$

for $k \geq K_w$, where $f(k, K_w) = \min(k - K_w, K_w)$. By substituting equation (12) into equation (4), we obtain the critical point, and equation (6) reveals that the critical temperature T_c does not exist other than at $K_w = 1$. In Figure 3b, the critical line $T-r$ which distinguishes chaotic and ordered states is given provided that $K_w = 1$. For a given T , the region $r < r_c$ is a chaotic state, whereas the region $r > r_c$ is an ordered state. We conclude that the larger the dispersion of $P(k)$, the smaller the regime of the chaotic state.

6 Conclusion

We have investigated the dynamic properties of the Threshold Network with thermal noise T on scale-free, random-graph, and small-world models. If no threshold $h = 0$ or thermal noise $T = 0$ exists, and if we have the rate $k = 0$ of the distribution degree $D = 0$, we find there is no phase transition with respect to average connectivity $\langle k \rangle$ in any network topologies. In the scale-free model, the minimum critical average connectivity $\langle k_c \rangle = 1$ is obtained provided that $h = 0$, $T = 0$, $D = 0$. In addition, the relationship between robustness and the stochasticity of the network has been clarified. Moderate thermal noise between the TN and the RBN extends the regime of ordered dynamics to higher average connectivity, i.e. suppresses the propagation of perturbation/damage, suggesting a new aspect to, for example, the theory of neural networks with particular network topologies. By tuning the thermal noise parameter in networks having asymmetric connections (which are usually found in neural networks) in which the basic unit (the neuron) follows threshold dynamics, the thermal noise, i.e. the non-zero error rate, may in fact improve information processing, given structure of the network topology. In this case, the thermal noise and network topology would be associated with the storage size, retrieval for patterns, or distribution of retrieval errors. In the organization of the human brain, the connection between neurons is not fully random, but rather appears to have a small-world topology, because in this type of neural network, the reduction in wiring costs [20] results in a level equivalent to that of *C. elegans* [2]. As a gene expression model, our model may provide an interesting starting point if further study reveals how the probability of gene expression depends statistically on the connectivity. Extending these ‘toy models’ may lead to a more realistic description of real systems.

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